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Araştırma Makalesi / Research Article

Novel One Parameter Family:Special Case, Bayesian Estimation, Simulation and Applications

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Abstract
This paper introduces a new one-parameter family (NPFD) derived from the cumulative distribution function (CDF). We study the main properties of the proposed family, with a special emphasis on its moments, reliability parameters, and asymptotic distributions of the extreme order statistics. Then, inferential considerations are explored. We discuss the parameter estimation by the moments, maximum likelihood methods and the Bayesian estimation. Also, likelihood estimation and Bayesian estimation using the Pitman asymptotic criterion are given. Three applications reveal that the new model can fit well practical data sets.

Keywords: New XLindley distribution, moments, reliability analysis, simulation

1. Introduction

Statistical models can be used to describe and predict real-world events. In recent years, many different types of distributions have been used to model data in many different fields. Recent advances have focused on establishing new families that extend well-known distributions while allowing great flexibility in modeling real-world data. Several distributions have been proposed in the statistical literature to modify lifespan data, including the Lindley (1958), Exponential, Gamma, Weibull, Zeghdoudi (Messaadia and Zeghdoudi, 2018), Xgamma (Sen et al., 2016), XLindley (Chouia and Zeghdoudi, 2021), New polynomial exponential (Beghriche et al., 2022), new XLindley (Khodja et al., 2023), ZLindley (2024) and squared ZLindley (Lazri et al., 2024) distributions. In this paper, we study a new one-parameter family ($NPFD$) that includes the new XLindley distribution in special case. Existing literature on survival data modeling, biosciences, and actuarial science will benefit from this new distribution group. Suppose T is a random variable whose values fall between [0, +∞] and whose distribution is dependent upon an unknown parameter θ has values within the range [0, +∞], and this is how its cumulative distribution function (CDF) is written,

$$F_{NPFD}(t;\theta) = 1\text{-}e^{-c(\theta)t}\,[a(\theta)\,\alpha(t)] \qquad (1)$$

where $a(\theta)$ and $c(\theta)$ are real-valued functions on [0, +∞] and $\alpha(t)$ depend on t and θ.

We can verify that the CDF is a right continuous function right away, and check $\alpha(t)$ for the necessary criteria to make f_{NPFD} a distribution.

The following is the format of this research paper:

Section 2 covers survival and hazard functions, moments, and other statistical properties. Sections 3 and 4 consider the estimation of maximum likelihood distribution parameters. Section 5 compares likelihood and Bayesian estimation using the Pitman asymptotic criterion. Finally, three specific applications demonstrate the superior performance of the new family of models (NPFD) over the exponential distributions, Lindley, Zeghdoudi, XLindley, Xgamma, and new XLindley.

2.Some Statistical Properties of NPFD

Proposition 1. The $F_{NPFD}(t;\theta)$in(1)of the NPFD is according to:

1. $F_{NPFD}(0,\theta)=0$ if $\alpha(0)=\frac{1}{\alpha(\theta)}$
2. $F_{NPFD}(\infty,\theta)=1$ if $\lim_{t\to\infty} a(t)e^{-c(\theta)t}=0$
3. $F_{NPFD}(t;\theta)$ increasing $if\,(c(\theta)-\alpha'(\theta))>0$

Proof.

1. We have: $F_{NPFD}(0,\theta)=1-a(\theta)a(0)$

By equating to zero and solving it with respect to t, we find: $\alpha(t)=\frac{1}{\alpha(\theta)}$

2. Since: $\lim_{t\to\infty} a(\theta)e^{-c(\theta)t}=0$, as well as being:

$$\lim_{t\to\infty} 1-a(\theta)e^{-c(\theta)t}-a(t)e^{-c(\theta)t}=1$$

we inserted $\lim_{t\to\infty} a(\theta)e^{-c(\theta)t} = 0$

3. The first derivatives of the *P D F* in (1) is determined as follows:

$$\frac{dF_{NPFD}}{dt} = a(\theta)e^{-c(\theta)t}[c(\theta)a(t) - \alpha'(t)] \qquad (2)$$

In order for $\frac{dF_{NPFD}}{dt}$ to be positive, we interpret as $(c(\theta)a(t) - \alpha'(t)) > 0$ (3).

2.1. Sub general cases

This family's inclusion of many selections of $a(t)$ that are documented in the literature s the ensuing a special cases are one of its most significant benefits, for examples:

1) $a(t) = s(t)$; with $s(t)$ is a survival function due to $s(t) = 1 - F(t)$ hence: $s'(t) = -f(t)$so$a'(t) < 0$
2) Decreasing fractional or polynomials functions.
3) By equating the linear differential equation of first order in Eq. (3) to zero and solving it with respect to *t*, we obtain the following solution:

$$a(t) = ke^{c(\theta)t}; k \in R$$

In this paper, we're going to study the case where$a(t) = S_{selected}(t)$; We get:

$$f_{N\,P\,F\,D}(t) = a(\theta)e^{-c(\theta)t}[c(\theta)s_s(t) + f_s(t)] \qquad (4)$$

A. Mode

This subsection addresses the *N P F D PDF's* shape properties in (4) at *t=0* and *t=∞*, respectively,

$$\lim_{t\to 0} f_{N\,P\,F\,D}(t) = c(\theta) + a(\theta)l(\theta) with\ l(\theta) = \lim_{t\to 0} f_s(t)$$

$$\operatorname{Lim}_{t\to\infty} = f_{NPFD}(t) = 0$$

Proposition 2.
The $PDFf(t;\theta)$ in (4) of the *N P F D* is decreasing if

$$c(\theta)[c(\theta)s_s(t) + 2f_s(t)] - \frac{df_s}{dt} > 0$$

And$\frac{d^2 f_s}{dt} > 0\ if c(\theta)^3 s_s(t) + 3c(\theta)^2 f_s(t) - 3c(\theta)\frac{df_s}{dt} + \frac{d^2 f_s}{dt} > 0$

Proof. The following is the determination of the *PDF's* first and second derivatives in equation (4):

$$\frac{df_{NPFD}}{dt} = -a(\theta)e^{-c(\theta)t}\left[c(\theta)\left[c(\theta)s_s(t) + 2f_s(t)\right] - \frac{df_s}{dt}\right]$$

$$\frac{d^2 f_{NPFD}}{dt^2} = a(\theta)e^{-c(\theta)t}\left[c(\theta)^3 s_s(t) + 3c(\theta)^2 f_s(t) - 3c(\theta)\frac{df_s}{dt} + \frac{d^2 f_s}{dt}\right]$$

B. Survival and hazard functions

The survival functions $S_{NPFD}(t)$ and hazard rate function *(hr f)* $h_{NPFD}(t)$ for the *N P D F* are, respectively, defined as follows:

$$S_{NPFD}(t) = 1 - F_{NPFD}(t) = a(\theta)s_s(t)e^{-c(\theta)t} \qquad (5)$$

$$h_{NPFD}(t) = \frac{f_{NPFD}(t)}{S_{NPFD}(t)} = c(\theta) + \frac{f_s(t)}{S_s(t)} = c(\theta) + h_s(t) \qquad (6)$$

C. Moments and related measures of the NPFD

Corollary1.

Let $T \sim NPFD$. Then, the i the moment of T is determined as follows:

$$\mu_p = E[T^\tau] = \int_0^\infty t^\tau f_{NPFD}(t)dt$$

$$= \int_0^\infty t^\tau a(\theta)e^{-c(\theta)t}[c(\theta)s_s(t) + f_s(t)]dt$$

$$= \omega(\theta) + \beta(\theta)\int_0^\infty \tau(\tau + 1; tc(\theta))[c(\theta)f_s(t) + f'_{s(t)}]\,dt$$

Where: $\int_0^\infty t^\tau e^{-c(\theta)t} = \frac{-1}{c(\theta)^{\tau+1}}\tau(\tau + 1; tc(\theta)) and \omega(\theta) = \frac{a(\theta)\tau(\tau+1)[c(\theta)s_s(0)+f_s(0)}{c(\theta)^{\tau+1}}$ and $\beta(\theta) = \frac{a(\theta)}{c(\theta)^{\tau+1}}$

Using Eq (7), one may calculate the first four moments of the *N P F D* random variable by entering the values $i = 1,2,3,4$. These moments are then used to calculate a number of statistical measures, including the coefficient of variation, skewness, kurtosis, and variance of *N P F D*, in that order:

$$Var\,[X] = E[X^2] - E[X]^2$$

Where $E[X^2] = \gamma(\theta)[\sigma(\theta) + \int_0^\infty \tau(3; tc(\theta))[c(\theta)f_s(t) + f'_{s(t)}]$

With $\gamma(\theta) = \frac{a(\theta)}{c(\theta)^3}$ and $\sigma(\theta) = 2[c(\theta)s_s(0) + f_s(0)]$

$$Skewness = \sqrt{\beta 1} = \frac{E[X^3]}{[Var(X)]^{\frac{3}{2}}}$$

Where $E[X^3] = k(\theta)[M(\theta) + \int_0^\infty \tau(4; tc(\theta))[c(\theta)f_s(t) + f'_{s(t)}]dt]$

With$k(\theta) = \frac{a(\theta)}{c(\theta)^4}$ and $M(\theta) = 6[c(\theta)s_s(0) + f_s(0)]$

$$kurtosis = B_2 = \frac{E[X^4]}{[Var\,[X]]^2}$$

Where $E[X^4] = A(\theta)[H(\theta) + \int_0^\infty \tau(5; tc(\theta))[c(\theta)f_s(t) + f'_{s(t)}]dt]$

With $A(\theta) = \frac{a(\theta)}{c(\theta)^5}$ and$H(\theta) = 120\,[c(\theta)s_s(0) + f_s(0)]$

3. Specific Case

As a specific example of (4), our suggested model is obtained as follows, We'll put the probability function (*pdf*) of new XLindley distribution (*N X L D)* (see Khodja et al. 2023) defined as:

$f_s(t) = f_{NXL}(x) = \frac{\theta}{2}(1+\theta x)e^{-\theta x},\ x\ \ ,\theta > 0$ and its related survival function $S_{NXL}(x)$ given by: $S_s(x) = S_{NXL}(x) = (\frac{1}{2}\theta x + 1)e^{-\theta x}$

With $a(\theta) = 1; c(\theta) = \theta$

We obtain;

$f_{NPFD}(x,\theta) = \frac{\theta}{2}(2\theta x + 3)e^{-2\theta x}x, \theta > 0$ (8)

Then the cumulative distribution function (*cdf*) of the *N P F D*:

$F_{NPFD}(x,\theta) = 1 - (\frac{1}{2}\theta x + 1)e^{-2\theta x}x, \theta > 0$ (9)

Therefore, the survival function $S_{NPFD}(x)$ and hazard rate function $h_{NPFD}(x)$ for the *N P FD* are respectively defined as follows:

$S_{NPFD}(x) = 1 - F_N(x,\theta) = (\frac{1}{2}\theta x + 1)e^{-2\theta x}x, \theta > 0$ (10)

$h_{NPFD}(x) = \theta + h_{NXL}(x) = \frac{3\theta + 2\theta^2 x}{\theta x + 2}x, \theta > 0$ (11)

Furthermore, the rth moment of the *N P D F* is defined as follows:

$$\mu_r = E[X^r] = \int_0^\infty x^r f_{N\,P\,F\,D}(x,\theta)dx = \int_0^\infty x^r \frac{\theta}{2}(2\theta x + 3)e^{-2\theta x}dx$$

$$= \frac{1}{4(2\theta)^r}[\tau(\tau + 2) + 3\tau(\tau + 1)] \qquad (12)$$

Where $\tau(z) = \int_0^\infty x^{z-1}e^{-x}dx$

Proposition3. Let X$\sim NPFD$, the mean, variance, coefficients of variation, skewness, and kurtosis for X are respectively defined as follows:

$E[X] = \frac{5}{8\theta}$, $Var[X] = \frac{23}{64\theta^2}$ Where $\tau(n) = (n-1)!$

$$Skewness = \sqrt{\beta 1} = \frac{E[X^3]}{[Var(X)]^{\frac{3}{2}}} = \frac{\frac{21}{16\theta^2}}{(\frac{3}{4\theta^2})^{\frac{3}{2}}} = \frac{7\sqrt[2]{3}}{6} = 2{,}0207$$

$$kurtosis = B_2 = \frac{E[X^4]}{[Var\ [X]]^2} = \frac{\frac{3}{\theta^4}}{(\frac{23}{64\theta^2})^2} = 23{,}2287$$

$$C.V = \theta = \frac{\sqrt[2]{Var\ (X)}}{E[X]} = \frac{\sqrt[2]{\frac{23}{64\theta^2}}}{\frac{5}{8\theta}} = \frac{\sqrt[2]{23}}{5}$$

The new distribution is leptokurtic and right-skewed according to the skewness and kurtosis.

***Theorem* 1.** Let X$\sim NPFD(\theta)$. Then the $median$ (X) $< E(X)$

Proof. Let m$\sim median(X)$ and $\mu = E(X) = \frac{5}{8\theta}$

Since the cumulative distribution function (c.d.f) is given by (9), it follows that $F(m) = \frac{1}{2}$

and$F(\mu) = 1 - \frac{21}{16}e^{-\frac{5}{4}}$

Note that $\frac{1}{2} < 1 - \frac{21}{16}e^{-\frac{5}{4}}$. Finally, since $F_{NPFD}(x)$ is an increasing function in $x > 0$ for all $\theta > 0$, we have $m < \mu$.

4. Estimation of the Unknown Parameters

In this part, we suggest analyzing the *N P D F* distribution in Eq. 8 using a Bayesian approach. For type *II* censored data, first, we offer the maximum likelihood (ML) estimation. Next, the Baysian estimation under the Linex, Entropy, and Generalized Quadratic (GQ) loss functions are discussed.

4.1. Maximum likelihood estimation

To estimate the parameter, we are interested in type *II* censored data. Assuming the *n*-sample ($x_1, x_2, \ldots, x_n$),i.e, and a constant *m*, we may sat that the *N P D F* distribution generates the *m*-sample ($x_1, x_2, \ldots, x_m$). The following is this sample's likelihood function:

For$n, m \in N$

$$L(\theta, X) = A \prod_{i=1}^{m} f_{N\,P\,F\,D}(x, \theta)[1 - F_{N\,P\,F\,D}(x_m)]^{n-m}$$

Where $A = \frac{n!}{(n-m)!}$

Replacing both (8) and (9) we have:

$$L(\theta, X) = A\left(\frac{\theta}{2}\right) n(\frac{\theta x_i}{2} + 1)^{n-m} e^{-2\theta(\sum_{i=1}^{m} x_i + (n+m)x_i)} \prod_{i=1}^{m}(2\theta x_i + 3) \qquad (13)$$

The equivalentlogarithmis :

$$l = l(x, \theta) = \ln L(\theta, X)$$

$$l = \ln A + n(\ln\theta - \ln 2) + (n-m)\ln(\frac{\theta x_i}{2} + 1) - 2\theta(\sum_{i=1}^{m} x_i + (n+m)x_i) + \sum_{i=1}^{m}(2\theta x_i + 3) \qquad (14)$$

The maximum likelihood estimator $\hat{\theta}_{MLE}$ of the parameter θ is obtained from the solution of the following non-linear system.

$$\hat{\theta}_{MLE} = \frac{\partial l}{\partial \theta} = \frac{n}{\theta} + (n-m)\frac{x_i}{\theta x_i + 2} - 2(n-m)x_i = 0 \qquad (15)$$

Since the system (15) solution appears to be intractable analytically, we will turn to numerical techniques to get approximate solution. Specifically, we will utilize the R package to derive the approximate value of the maximum likelihood estimator $\hat{\theta}_{MLE}$of the parameterθ.

A. Bayesian estimation

We discuss the Bayesian estimation in this part. In this approach, we resume a prior distribution of the parameter to be estimated based on a piece of prior information, treating the unknown values as random variables. We make use of the non-informative form of prior distribution for the parameterθ.

$$\pi(\theta) = \frac{1}{\theta}$$

The prior distribution is:

$$\pi(\theta/X) = \frac{\pi(\theta)L(\theta, X)}{\int_0^{\infty} \pi(\theta)L(\theta, X)d\theta}$$

When estimating using Bayesian methods for type *II* censored data, we additionally use Eq (13) to read the posterior distribution, which is as follows:

$$\pi(\theta/X) = \frac{K}{\theta}\left(\frac{\theta}{2}\right) n(\frac{\theta x_i}{2} + 1)^{n-m} e^{-2\theta(\sum_{i=1}^{m} x_i + (n+m)x_i)} \prod_{i=1}^{m}(2\theta x_i + 3) \qquad (14)$$

Where:

$$k=\frac{1}{\int_0^\infty \frac{K}{\theta}\left(\frac{\theta}{2}\right)n(\frac{\theta x_i}{2}+1)^{n-m}e^{-2\theta(\sum_{i=1}^m x_i+(n+m)x_i)}\prod_{i=1}^m(2\theta x_i+3)}$$

B. Estimators and their corresponding risks

The three loss functions: Entropy, Generalized Quadratic, and Linex are described in the table below.

loss function expression	Bayes estimators	posterior risk
Entropy: $L(\theta,\delta)=(\frac{\delta}{\theta})^p-p\,log\left(\frac{\delta}{\theta}\right)-1$	$\hat{\delta}_E=E_\pi(\theta-p)^{\frac{-1}{p}}$	$p[E_\pi\,(\log\theta-\log(\hat{\delta}_E)))]$
Generalized quadratic: $L(\theta,\delta)=\tau(\theta)(\theta-\delta)^2$	$\hat{\delta}_{GQ}=\frac{E_\pi(\tau(\theta)\theta)}{E_\pi(\tau(\theta))}$	$E_\pi(\tau(\theta)(\theta-\delta)^2)$
Linex :$L(\theta,\delta)=\exp\big(r(\delta-\theta)\big)-r(\delta-\theta)-1$	$\hat{\delta}_L=-\frac{1}{r}\log(E_\pi(\exp(-r\theta))$	$r(\hat{\delta}_{GQ}-\hat{\delta}_L)$

(1) We obtain the estimator and its corresponding risk (where *p* is an integer)
Under the Entropy loss function:

$$\hat{\theta}_E=\left[\int_0^\infty \theta^{-p}\pi(\theta/X)\,d\theta\right]^{-\frac{1}{p}}$$

$$\hat{\theta}_E=\left[\frac{K}{2^n}\int_0^\infty \theta^{n-p-1}(\frac{\theta x_i}{2}+1)^{n-m}e^{-2\theta(\sum_{i=1}^m x_i+(n+m)x_i)}\prod_{i=1}^m(2\theta x_i+3)\,f\theta\right]^{-\frac{1}{p}}$$

$$PR\big(\hat{\theta}_{GQ}\big)=p[E_\pi(\log(\theta-\log(\hat{\delta}_E)))]$$

(2) We obtain the estimator and its corresponding risk (where $\tau(\theta)=\theta^{\gamma-1}$, $\gamma$is an integer)under the Generalized quadraticloss function:

$$\hat{\theta}_{GQ}=\frac{\int_0^\infty \theta^\alpha\pi(\theta/X)d\theta}{\int_0^\infty \theta^{\alpha-1}\pi(\theta/X)d\theta}$$

$$\hat{\theta}_{GQ}=\frac{\int_0^\infty \theta^{\alpha+n-1}(\frac{\theta x_i}{2}+1)^{n-m}e^{-2\theta(\sum_{i=1}^m x_i+(n+m)x_i)}\prod_{i=1}^m(2\theta x_i+3)\,d\theta}{\int_0^\infty \theta^{\alpha+n-2}(\frac{\theta x_i}{2}+1)^{n-m}e^{-2\theta(\sum_{i=1}^m x_i+(n+m)x_i)}\prod_{i=1}^m(2\theta x_i+3)\,d\theta}$$

$$PR\big(\hat{\theta}_{GQ}\big)=E_\pi(\theta^{\gamma+1})-2\hat{\theta}_{GQ}E_\pi(\theta^\gamma)+\hat{\theta}^2{}_{GQ}E_\pi(\theta^{\gamma-1})$$

(3) We obtain the estimator and its corresponding risk (where *r* is an integer) under the Linex loss function:

$$\hat{\theta}_L=-\frac{1}{r}\log\left[\int_0^\infty e^{-r\theta}\pi(\theta/X)\,d\theta\right]$$

$$\hat{\theta}_L=-\frac{1}{r}\log[\frac{K}{2^n}\int_0^\infty \theta^{n-1}(\frac{\theta x_i}{2}+1)^{n-m}e^{-\theta(r+2(\sum_{i=1}^m x_i+(n+m)x_i))}\prod_{i=1}^m(2\theta x_i+3)\,d\theta]$$

$$PR\big(\hat{\theta}_L\big)=r(\hat{\theta}_{GQ}-\hat{\theta}_L)$$

5. Comparing the likelihood estimation and the Bayesian estimation using Pitman’s closeness criterion

In order to compare the performance of the proposed Bayes estimators with the MLEs, we perform a Monte Carlo simulation study assuming that$\beta = 1{,}5$, and using $N = 5000$ samples of the type II censored model with different sample sizes $n = 10, 50, 200,$ while $m = 8, 40, 160$ respectively, we obtain the following results. Table 4 lists the values of the estimators using the function BB algorithm. We remark here that the estimated values of β are close to the true values of the parameter especially with the increase in sample size$n$. Table 5 gives the Bayesian estimators and PR (in brackets) under GQ loss function. Table 6 presents the Bayesian estimators and PR (in brackets) under the entropy loss function. Table 7 gives Bayesian estimators and PR (in brackets) underLinex loss function. Table 8 shows the Bayesian estimators and PR (in brackets) under the three loss functions. In Table 5, the estimation under the GQ loss function, we remark that the value$\gamma = 1$one gives the best posterior risk. Also, we obtain the smallest suitable posterior risk when n is high. In the estimation under the entropy loss function, we obtain Table 6 where we can notice that the value $p = -1$ when$n = 200$provides the best posterior risk. We can notice clearly that the value $r = 1$provides the best PR Summing up, making a small comparison between the three loss functions, it is clear that the best results are obtained by the quadratic loss function, Table 8 illustrate those results in details. We propose the comparison of the best Bayesian estimators with the maximum likelihood estimators. For this purpose, we use the Pitman closeness criterion (see Pitman (1937), Fuller (1982) and Jozani (2012) for more details).

Table 1. The MLE of the parameters with quadratic error (in brackets)

$N = 5000$	$n = 10$	$n = 50$	$n = 200$
m	8	40	160
β	0.6235(0.0056)	0.8389(0.0044)	0.9675(0.00223)

Table 2: Bayes estimators and PR (in brackets) under GQ loss function

γ	$N = 5000$	$n = 10$	$n = 50$	$n = 200$
	m	8	40	160
$\lambda = -2$	β	1,342(0.0031)	1.4632(0.0021)	1.4743(0.0032)
$\lambda = -1{,}5$	β	1.321(0.0025)	1.3839(0.0021)	1.6926(0.0032)
$\lambda = -1$	β	1.3998(0.0031)	1.4213(0.0070)	1.3421(0.0018)
$\lambda = -0{,}5$	β	1.4768(0.1241)	1.5158(0.0033)	1.4991(0.1181)
$\lambda = 0{,}5$	β	1.7990(0.0087)	1.0825(0.0061)	1.2127(0.0016)
$\lambda = 1$	β	1.4999(0.0534)	1.4705(0.711)	1.5012(0.0012)
$\lambda = 1{,}5$	β	1.6132(0.0012)	1.4308(0.0070)	1.3412(0.0021)
$\lambda = 2$	β	1.2732(0.1004)	1.5711(0.1231)	1.6903(0.0003)

Table 3. Bayes estimators and PR (in brackets) under the entropy loss function.

γ	$N = 5000$	$n = 10$	$n = 50$	$n = 200$
	m	8	40	160
$p = -2$	β	11232(0.0042)	1.5632(0.0081)	1.6743(0.0098)
$p = -1{,}5$	β	1.7510(0.0095)	2.1839(0.0020)	1.7926(0.0077)
$p = -1$	β	1.0994(0.0089)	1.0888(0.0070)	1.2138(0.0018)
$p = -0{,}5$	β	1.4768(0.1241)	1.5158(0.0033)	1.4991(0.1181)
$p = 0{,}5$	β	1.7990(0.0087)	1.0825(0.0061)	1.2127(0.0016)
$p = 1$	β	1.2999(0.0825)	1.2701(0.711)	1.6432(0.0016)
$p = 1{,}5$	β	1.7131(0.0012)	1.0888(0.0070)	1.6432(0.0016)
$p = 2$	β	1.4768(0.1241)	1.6754(0.1181)	1.7903(0.0033)

Table 4. Bayes estimators and PR (in brackets) under Linex loss function

r	$N = 5000$	$n = 10$	$n = 50$	$n = 200$
	m	8	40	160
..$= -2$	β	1.3188(0.0699)	1.2839(0.009)	0.7034(0.011)
..$= -1{,}5$	β	1.4407(0.0611)	1.4077(0.0661)	1.7060(0.0012)
..$= -1$	β	1.4177(0.0072)	1.3633(0.0073)	0.7051(0.0003)
..$= -0{,}5$	β	0.6493(0.0308)	07037(0.0009)	0.8755(0.319)
..$= 0{,}5$	β	1.8895(0.0729)	1.8998(0.0008)	1.9814(0.0001)
..$= 1$	β	1.4148(0.0009)	1.4981(0.0038)	1.5100(0.0733)
..$= 1{,}5$	β	1,6037(0.0009)	1.3055(0.319)	1.5491(0.0308)
..$= 2$	β	1.4239(0.0199)	1.3881(0.0303)	1.7059(0.0003)

Table 5. Bayes estimators and PR (in brackets) under the three loss function

	$N = 5000$	$n = 10$	$n = 50$	$n = 200$
	m	8	40	160
$GQ\|_{\gamma=1}$	β	1.4999(0.0534)	1.4705(0.711)	1.5012(0.0012)
$Entropy\|_{p=0.5}$	β	1.4768(0.1241)	1.5158(0.0033)	1.4991(0.1181)
$Linex\|_{r=1.5}$	β	1.4148(0.0009)	1.4981(0.0038)	1.5100(0.0733)

Definition 1: An estimator θ_1 of a parameter θ dominates another estimator θ_2 in the sense of Pitman's closeness criterion if for all $\theta \in \Phi$

$$P_\theta[|\theta_1 - \theta| < |\theta_2 - \theta|] > 0.5$$

In Table 9, we present the values of the Pitman probabilities which allows us to compare the Bayesian estimators with the MLE estimator which is done under the three loss functions when $\gamma = 1,\ p = 0.5\ ,\ r = 1.5$. According to definition 1, when the probability is greater than 0.5, the Bayesian estimators are better than the MLE estimators. Then we notice that, according to this criterion, the Bayesian estimators of the parameters is better than the MLE. Also the GQ loss function has the best values in comparison with the other two loss functions with

$$a = 0.745|_{n=10,m=10},\ 0.744|_{n=50,m=40} \text{ and } 0.798|_{n=200,m=160}.$$

Table 6: Pitman comparison of the estimators

	$N = 5000$	$n = 10$	$n = 50$	$n = 200$
	m	8	40	160
$GQ\|_{\gamma=1}$	β	0.745	0.744	0.798
$Entropy\|_{p=0.5}$	β	0.656	0.582	0.567
$Linex\|_{r=1.5}$	β	0.712	0.544	0.544

6. Application with Real Data Set

Three applications are now proposed to illustrate the usefulness of the proposed distribution. More precisely, we explore the tuning behavior of the NPFD compared to the exponential, the Lindley, Zeghdoudi, XLindley, Xgamma and new XLindley distributions. For this, we estimate the unknown parameters of the respective model using the maximum likelihood method and consider their corresponding standard errors (SE), the estimated log likelihoods (- 2logL), the values of AIC (Akaike information criterion) AICC (Akaike information criterion correction), HQIC (Hannan–Quinn information criterion) and BIC (Bayesian information criterion).

Data set 1: For numbers of users connected to the Internet data

A time series of the numbers of users connected to the Internet through a server every minute. See (Durbin and Koopman (2001))
88,84,85,85,84,85,83,85,88,89,91,99,104,112,126,138,146,151,150,148,147,149,143,132,131,139,147,150,148,145,140,134,131

,131,129,126,126,132,137,140,142,150,159,167,170,171,172,172,174,175,172,172,174,174,169,165,156,142,131,121,112,104,102,99,99,95,88,84,84,87,89,88,85,86,89,91,91,94,101,110,121,135,145,149,156,165,171,175,177,182,193,204,208,210,215,222,228,226,222,220.

Model	θ	AIC	BIC	-2L	AICC	HQIC
Exponential	0.007305803	1186.113	1188.718	1184.113	1186.154	1187.168
Lindley	0.01450447	1118.71	1121.315	1116.71	1118.751	1119.764
XLindley	0.01439125	1119.983	1122.589	1117.983	1120.024	1121.038
New-XLindley	0.01172219	1162.795	1165.4	1160.795	1162.835	1163.849
Xgamma	0.02168419	1085.733	1088.338	1083.733	1085.773	1086.787
Zeghdoudi	0.02180811	1083.269	1085.874	1081.269	1083.31	1084.323
NPFD	0.00472895	1179.935	1182.54	1177.935	1179.976	1180.989

7. For Yearly Numbers of Important Discoveries data

The numbers of "great" inventions and scientific discoveries in each year from 1860 to 1959. See (The World Almanac and Book of Facts, 1975) 5, 3, 0, 2, 0, 3, 2, 3, 6, 1, 2, 1, 2, 1, 3, 3, 3, 5, 2, 4, 4, 0, 2, 3, 7, 12, 3, 10, 9, 2, 3, 7, 7, 2, 3, 3, 6, 2, 4, 3, 5, 2, 2, 4, 0, 4, 2, 5, 2, 3, 3, 6, 5, 8, 3, 6, 6, 0, 5, 2, 2, 2, 6, 3, 4, 4, 2, 2, 4, 7, 5, 3, 3, 0, 2, 2, 2, 1, 3, 4, 2, 2, 1, 1, 1, 2, 1, 4, 4, 3, 2, 1, 4, 1, 1, 1, 0, 0, 2, 0

Model	θ	AIC	BIC	-2L	AICC	HQIC
Exponential	0.3225761	428.2804	430.8856	426.2804	428.3212	429.3348
Lindley	0.5330051	417.7608	420.366	415.7608	417.8016	418.8152
XLindley	0.4703674	420.9874	423.5926	418.9874	421.0283	422.0418
New-XLindley	0.4922593	420.1228	422.728	418.1228	420.1636	421.1772
Xgamma	0.7153558	415.2443	417.8495	413.2443	415.2852	416.2987
Zeghdoudi						
NPFD	0.2045404	425.613	428.2182	423.613	425.6538	426.6673

8. Populations Recorded by the US Census data

This data set gives the population of the United States (in millions) as recorded by the decennial census for the period 1790--1970. See (McNeil (1977)**).** 3.93, 5.31, 7.24, 9.64, 12.90, 17.10, 23.20, 31.40, 39.80, 50.20, 62.90, 76.00, 92.00, 105.70, 122.80, 131.70, 151.30, 179.30, 203.20

Model	θ	AIC	BIC	-2L	AICC	HQIC
Exponential	0.01435232	201.3175	202.2619	199.3175	201.5528	201.4773
Lindley	0.0282759	207.6266	208.571	205.6266	207.8619	207.7864
XLindley	0.02790613	206.924	207.8684	204.924	207.1593	207.0838
New-XLindley	0.02114924	201.6523	202.5968	199.6523	201.8876	201.8122
Xgamma	0.04029827	215.4385	216.3829	213.4385	215.6738	215.5983
Zeghdoudi	0.04269358	220.9815	221.926	218.9815	221.2168	221.1414
NPFD	0.008958483	201.2335	202.1779	199.2335	201.4688	201.3933

9. Conclusion and Perspectives

In this paper we have shown how probability distributions can be constructed without adding additional parameters or using the usual generalizations techniques. The proposed distribution is called the NPFD. It can be seen that the NPFD has many desirable properties. We have derived precise and explicit expressions for many characteristics, in particular moments, reliability parameters and asymptotic distributions of order statistics. For estimating parameters, we have discussed the method of moments and the method of maximum likelihood. In addition, NPFD, exponential, Lindley, Zeghdoudi, XLindley, Xgamma and new XLindley distributions were fitted to three real datasets; and the results showed that the NPFD distribution is a strong candidate with one parameter. We can also use the NPFD distribution as the basis for new distributions, for other distributions from a statistical perspective, such as the pioneering work of Beghriche et al. (2022).

Declaration of Author Contributions

The authors declare that they have contributed equally to the article. All authors declare that they have seen/read and approved the final version of the article ready for publication.

Declaration of Conflicts of Interest

All authors declare that there is no conflict of interest related to this article.

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