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Square ZLindley Distribution:Statistical Properties, Simulation and Applications in Sciences

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Abstract

An extension of the current ZLindley distribution, the new one-parameter Square ZLindley distribution (SZLD) is presented in this paper. It is possible to utilise the suggested model with both left-symmetric and left-skewed data sets. The shape of the SZLD will be discussed. Additionally included are quantile functions, moment generation functions, mean lifespan functions, Rényi entropy, order statistics, and survival and hazard functions. To effectively convey the usefulness of the suggested distribution, statistical features like moments, modes, quantile functions, and moment generator functions are produced. Using the maximum likelihood estimation method, parameters were computed. A comprehensive simulation analysis is conducted to assess these suggested estimators' performance using MLE for various parameter values two real-world datasets are used to illustrate the applicability and flexibility of the newly suggested distribution. Additional statistical inferences on the SZLD are supplied by data fitting, simulation studies, and graphing, utilising R and Maple tools.

Keywords: ZLindley distribution, moments, reliability analysis, simulation

1. Introduction

In many fields, including as actuarial science, management, engineering, physics, biology, hydrology, and computer science, the modelling and analysis of lifetime data is essential. Traditional probability distributions have been applied to a range of data sizes. When classical or traditional probability models are unable to adequately describe real-world data, significant issues occur. Consequently, it is imperative to enhance the flexibility of existing probability models by adding more parameters or combining two distributions. Various approaches to include components in the primary model have been recorded in academic research. Surveys conducted by Azzalini (1985), Bourguignon et al. (2014), Eugene et al. (2002) and Shawet al. (2009) are recommended for a comprehensive analysis of generalisation methods used on baseline distributions. These websites are excellent sources for additional research and provide insightful analyses on the topic. The statistical literature has suggested a number of one-parameter models, such as the Lindley, exponential, Zeghdoudi, Shanker, XLindley, new XLindley, and Xgamma, to alter lifespan data. The purpose of this article is, firstly, to propose and study a new distribution with one parameter using the square transformation. It may be used in a wide variety of areas, including biology, engineering, astronomy, actuarial science, and medicine. On the other hand, the new distribution has an increased risk rate and a decreasing average residual life function. This new distribution may attract research attention. The new one-parameter polynomial exponential distribution (NPED) is a new statistical family that was recently introduced by Beghriche et al. (2022) and Saaidia et al. (2024). The probability density function (p.d.f) of this distribution defines.

$$f_{NPED}(\mathrm{t},\theta)=\frac{P(t,\theta)e^{-\theta t}}{\sum_{k=0}^{n}a_{k,\theta}\frac{k!}{\theta^{k+1}}}\quad ;t,\theta>0 \qquad (1)$$

Where, $P(t,\theta)=\sum_{k=0}^{n}a_{k,\theta}t^{k}$, and $a_{k,\theta}$ depend on $k$and θ.

The ZLindley distribution (ZLD) is derived as a specific instance of equation (1), when $n=1, a_{0,\theta}=1+2\theta, a_{1,\theta}=\theta$as follows

$$f(t)=\frac{\theta}{2(1+\theta)}(1+2\theta+\theta t)e^{-\theta t},\quad t,\theta>0$$

The following are the main objectives of our study: The objective is to identify its qualities and provide a new enlargement of the Square ZLindley distribution. to use the maximum likelihood estimation technique to determine the parameters of the new distribution. Applying the suggested distribution to two real datasets.

2. Derivation of the Square ZLindley Model

The Square ZLindley distribution(SZLD) is obtained by applying a following transformation $Y=T^{\frac{1}{2}}$, where T follows the ZLindley distribution.The random variable Y is distributed according to the SZLD, which is symbolically represented as $Y\sim SZLD(\theta)$. The cumulative distribution function (CDF) of the SZLD is obtained in the following manner:

$$F(y) = 1 - \left(1 + \frac{\theta y^2}{2(1+\theta)}\right) e^{-\theta y^2}, \tag{2}$$

The corresponding probability density function (PDF) is

$$f(y) = \frac{\theta y(1 + 2\theta + \theta y^2)}{(1+\theta)} e^{-\theta y^2}. \tag{3}$$

2.1. Shape of SZLD

The limiting behavior of PDF at the upper limit is given in the following Theorem 1. The limiting behavior of PDF at the lower limit is

$$\lim_{y\to 0} f(y) = 0.$$

The limiting pattern at the upper limit is

$$\lim_{y\to \infty} f(y) = 0.$$

Theorem 1. *The density function of the SZLD is unimodal behavior start from zero where the mode is*

$$Mode = 2^{-\frac{1}{2}}\left\{\frac{-(4\theta-1)+\sqrt{(9+8\theta+16\theta^2)}}{2\theta}\right\}^{\frac{1}{2}}.$$

Proof. *Firstly, we determine the first derivative of the PDF for Y of the SZLD as follows:*

$$\frac{\partial f(y)}{\partial y} = \frac{\theta\left(-2\theta^2 y^4 - y^2(4\theta^2 - \theta) + (1 + 2\theta)\right)e^{-\theta y^2}}{(1+\theta)},$$

Then equating the previous equation to zero and solving if for y, we have the SZLD modes as follows: .

$$Mode = 2^{-\frac{1}{2}}\left\{\frac{-(4\theta-1)+\sqrt{(9+8\theta+16\theta^2)}}{2\theta}\right\}^{\frac{1}{2}}, and \ \frac{\partial^2 f(mode)}{\partial y^2} < 0.$$

Since, The PDF of the SZLD is unimodal behavior start from zero.

3. Statistical Properties

The main goal of this section is to derive various statistical properties of the proposed distribution. The moments and related measures, quantile function, mean residual life function, actuarial measure, and order statistics are some examples of these characteristics.

3.1. Moments

The rth moments of SZLD are

$$\mu_r' = \int_0^\infty \frac{\theta y^{r+1}(1+2\theta+\theta y^2)}{(1+\theta)} e^{-\theta y^2}\, dy$$

$$\mu_r' = \frac{\theta^{-\frac{r}{2}}\left(r+4(1+\theta)\right)\Gamma\left(1+\frac{r}{2}\right)}{4(1+\theta)}. \tag{4}$$

The first four moments about the origin are obtained by taking $r = 1,2,3,$ and 4.

$$\mu_1' = \frac{\theta^{-\frac{1}{2}}(5+4\theta)\Gamma\left(\frac{3}{2}\right)}{4(1+\theta)}. \tag{5}$$

$$\mu_2' = \frac{\theta^{-1}(6+4\theta)\Gamma(2)}{4(1+\theta)}. \tag{6}$$

$$\mu_3' = \frac{\theta^{-\frac{3}{2}}(7+4\theta)\Gamma\left(\frac{5}{2}\right)}{4(1+\theta)}. \tag{7}$$

$$\mu_4' = \frac{\theta^{-2}(8+4\theta)\Gamma(3)}{4(1+\theta)}. \tag{8}$$

Additionally, by using moments around the origin, we can determine the ith central moments of Y as follows:

$$\mu_i = E(y-\mu)^i = \sum_{m=0}^{\infty} (-1)^m \binom{i}{m} \mu_1'^m \mu_{i-m}'.$$

$$E(X) = \mu_1' = \frac{\theta^{-\frac{1}{2}}(5+4\theta)\Gamma\left(\frac{3}{2}\right)}{4(1+\theta)}$$

and

$$VAR(X) = \mu_2' - (\mu_1')^2 = \frac{4(6+4\theta)(1+\theta) + (5+4\theta)^2\Gamma\left(\frac{3}{2}\right)^2}{16\theta(1+\theta)^2}$$

The formulas for calculating the coefficient of variance, coefficient of skewness, and kurtosis can be obtained using the following equations.

$$CV = \frac{SD(x)}{Mean(x)} = \frac{\sqrt{Var(X)}}{E(X)} = \frac{\sqrt{4(6+4\theta)(1+\theta) + (5+4\theta)^2\Gamma\left(\frac{3}{2}\right)^2}}{(5+4\theta)\Gamma\left(\frac{3}{2}\right)}$$

$$\gamma_1 = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}{(\mu_2' - (\mu_1')^2)^{\frac{3}{2}}} = \frac{E(X^3)}{(Var(X))^{\frac{3}{2}}} = \frac{16(1+\theta)^2(7+4\theta)\Gamma\left(\frac{5}{2}\right)}{\left(4(6+4\theta)(1+\theta)+(5+4\theta)^2\Gamma\left(\frac{3}{2}\right)^2\right)^{\frac{3}{2}}}$$

and

$$\gamma_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4}{(\mu_2' - (\mu_1')^2)^2} = \frac{E(X^4)}{\left(Var(X)\right)^2} = \frac{128(8+4\theta)(1+\theta)^3}{\left(4(6+4\theta)(1+\theta)+(5+4\theta)^2\Gamma\left(\frac{3}{2}\right)^2\right)^2}$$

Table 1 provides certain numerical values 530ort he mean, variance, coefficient of skewness, and coefficient of kurtosis for a few chosen parameter values.

Table 1. Some computational statistics of SZL distribution

θ	Mean	Var	CV	Skewness	Kurtosis
0.10	3. 4394	26. 375	1. 4932	0.52194	0.54887
0.50	1. 4622	4. 8047	1. 4991	0.53552	0.57757
1. 0	0.99701	2. 244	1. 5025	0.54375	0.59576
2.0	0.67888	1. 0442	1. 5052	0.55058	0.61141

3.2. Quantile Function

As can be shown from Eq. (2), the quantile function F is continuous and strictly increasing, and is thus defined as

$$Quantile = y_p = F_Y^{-1}(p) \qquad (9)$$

Theorem 2. *For $p = F_Y(y)$, using the Lambert W function the expression 530ort he quantile function is obtainable, for any $\theta > 0$, the y_p of SZLD is:*

$$y_p = \left\{-\frac{2(1+\theta)}{\theta} - \frac{1}{\theta} W_{-1}\left[(p-1)2(1+\theta)e^{-2(1+\theta)}\right]\right\}^{\frac{1}{2}} \qquad (10)$$

where W_{-1} is the negative branch of the Lambert W function.

Proof: For any $\theta > 0$, suppose 0 < P < 1

From cdf of SZLD, The equation $p = F_Y(y)$ will be solved for y.

$$1 - \left(1 + \frac{y^2\theta}{2(1+\theta)}\right)e^{-\theta y^2} = p$$

$$[2(1+\theta) + y^2\theta]e^{-\theta y^2} = 2(1-p)(1+\theta) \qquad (11)$$

Multiply both sides of Eq. (11) by $e^{-2(1+\theta)}$ we get

$$[2(1+\theta)+y^2\theta]e^{-\theta y^2}e^{-2(1+\theta)} = 2(1-p)(1+\theta)e^{-2(1+\theta)}$$

$$-[2(1+\theta)+y^2\theta]e^{-[2(1+\theta)+\theta y^2]} = 2(\mathrm{p}-1)(1+\theta)e^{-2(1+\theta)}$$

Applying the Lambert W function on both sides we get

$$W\left[-[2(1+\theta)+y^2\theta]e^{-[2(1+\theta)+\theta y^2]}\right] = W\left[2(\mathrm{p}-1)(1+\theta)e^{-2(1+\theta)}\right] \quad (12)$$

As we know that $W[ze^z] = z$, then Eq. (10) becomes

$$-[2(1+\theta)+\theta y^2] = W\left[2(\mathrm{p}-1)(1+\theta)e^{-2(1+\theta)}\right]$$

For $\theta > 0$ and$y > 0, 2(1+\theta)+y^2\theta > 0$ and it is also checked $2(\mathrm{p}-1)(1+\theta)e^{-2(1+\theta)} \in \left(\frac{1}{e}, 0\right)$ since 0 < p <

3. Thus, by using the properties of the negative branch W_{-1} of the Lambert W function. Hence

$$-[2(1+\theta)+y^2\theta] = W_{-1}\left[2(\mathrm{p}-1)(1+\theta)e^{-2(1+\theta)}\right]$$

$$y_p = \left\{-\frac{2(1+\theta)}{\theta} - \frac{1}{\theta}W_{-1}\left[2(\mathrm{p}-1)(1+\theta)e^{-2(1+\theta)}\right]\right\}^{\frac{1}{2}}.$$

Table 2 shows some quantiles of the SZLD, which have been calculated from the closed form expression for $F_Y^{-1}(p)$

p	$\theta = 0.01$	$\theta = 0.1$	$\theta = 1.5$	$\theta = 3$
0.01	1. 4074	0.42851	9. 1502×10⁻²	6. 1873×10⁻²
0.05	3. 1506	0.96157	0.20658	0.13975
0.1	4. 4675	1. 3671	0.29583	0.20024
0.25	7. 1738	2. 2092	0.48756	0.33058
0.4	9. 3191	2. 8834	0.64776	0.44005
0.5	10. 681	3. 3136	0.75285	0.51218
0.6	12. 084	3. 7578	0.86337	0.58831
0.75	14. 487	4. 5201	1. 0568	0.72226
0.9	18. 067	5. 6574	1. 3514	0.92778

3.3. Moment Generating Function

The moment-generating function is derived as follows

$$M_Y(t) = E(e^{ty}) = \int_0^\infty e^{ty} f(y)dy$$

$$M_Y(t) = \int_0^\infty e^{ty} \frac{\theta y(1+2\theta+\theta y^2)}{(1+\theta)} e^{-\theta y^2} dy \qquad (13)$$

We know that $e^{ty} = \sum_{r=0}^{\infty} \frac{(ty)^r}{r!}$

Eq. (13) will be

$$M_Y(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\frac{\theta^{-\frac{r}{2}} \left(r + 4(1+\theta)\right) \Gamma\left(1+\frac{r}{2}\right)}{4(1+\theta)} \right). \qquad (14)$$

3.4. Rényi Entropy

Rényi entropy can be derived as

$$H_R(\mathrm{H}) = \frac{1}{1-\mathrm{H}} \ln \left[\int_{-\infty}^{\infty} (f(y))^\eta dy \right], \eta > 0, \eta \neq 1 \qquad (15)$$

When Y~ SZLD, then

$$\int_0^\infty (f(y))^\eta dy = \int_0^\infty \left[\frac{\theta y(1+2\theta+\theta y^2)}{(1+\theta)} e^{-\theta y^2} \right]^\eta dy$$

$$\int_0^\infty (f(y))^\eta dy = \frac{\theta^\eta (1+2\theta)^\eta}{(1+\theta)^\eta} \int_0^\infty \left(1 + \frac{\theta y^2}{1+2\theta}\right)^\eta y^\eta e^{-\theta\eta y^2} dy$$

Using the following binomial expansion

$$(1+z)^n = \sum_{i=1}^{\infty} \binom{n}{i} z^i$$

$$\left(1 + \frac{\theta y^2}{1+2\theta}\right)^\eta = \sum_{i=1}^{\infty} \binom{\eta}{i} \left(\frac{\theta y^2}{1+2\theta}\right)^i = \sum_{i=1}^{\infty} \binom{\eta}{i} \frac{\theta^i}{(1+2\theta)^i} y^{2i}$$

$$\int_0^\infty (f(y))^\eta dy = \frac{\theta^\eta (1+2\theta)^\eta}{(1+\theta)^\eta} \sum_{i=1}^{\infty} \binom{\eta}{i} \frac{\theta^i}{(1+2\theta)^i} \int_0^\infty y^{2i+\eta} e^{-\theta\eta y^2} dy$$

Substitute$\theta\eta y^2 = t$

$$if \quad y \to 0 \; then \qquad t \to 0 \;\; and \; y \to \infty \qquad then \qquad t \to \infty$$

$$y^2 = \frac{t}{\theta\eta} \quad and \quad y = \left(\frac{t}{\theta\eta}\right)^{\frac{1}{2}} and \;\; also \quad 2\theta\eta y \, dy = dt$$

$$dy = \frac{dt}{2\theta\eta y} = \frac{dt}{2\theta\eta \left(\frac{t}{\theta\eta}\right)^{\frac{1}{2}}}$$

$$\int_0^\infty (f(y))^\eta dy = \frac{\theta^\eta(1+2\theta)^\eta}{(1+\theta)^\eta}\sum_{i=1}^{\infty}\binom{\eta}{i}\frac{\theta^i}{(1+2\theta)^i}\frac{1}{2(\theta\eta)^{\frac{2i+\eta}{2}+\frac{1}{2}}}\int_0^\infty t^{\left(\frac{2i+\eta}{2}-\frac{1}{2}\right)}e^{-t}\,dt$$

$$\int_0^\infty (f(y))^\eta dy = \frac{\theta^\eta(1+2\theta)^\eta}{2(1+\theta)^\eta}\sum_{i=1}^{\infty}\binom{\eta}{i}\frac{\theta^i}{(1+2\theta)^i}\frac{1}{(\theta\eta)^{\frac{\eta+1}{2}+i}}\Gamma\left(\frac{2i+\eta+1}{2}\right)$$

Hence Eq. (15) becomes

$$H_R(\mathrm{H}) = \frac{1}{1-\mathrm{H}}\ln\left\{\frac{\theta^\eta(1+2\theta)^\eta}{2(1+\theta)^\eta}\sum_{i=1}^{\infty}\binom{\eta}{i}\frac{\theta^i}{(1+2\theta)^i}\frac{1}{(\theta\eta)^{\frac{\eta+1}{2}+i}}\Gamma\left(\frac{\eta+1}{2}+i\right)\right\} \qquad (16)$$

3.6. Survival and Hazard function

The survival function of SZLD is

$$S(y) = \left(1+\frac{\theta y^2}{2(1+\theta)}\right)e^{-\theta y^2}, \qquad (17)$$

The hazard function is obtained as

$$h(y) = \frac{2\theta y(1+2\theta+\theta y^2)}{2+2\theta+\theta y^2}. \qquad (18)$$

3.7. Mean Residual Life Function

$$m(t) = \frac{1}{S(t)}\int_t^\infty y f(y)\,dy - t \qquad (19)$$

Consider an integral part

$$\int_t^\infty y f(y)\,dy = \int_t^\infty \frac{\theta y^2(1+2\theta+\theta y^2)}{(1+\theta)}e^{-\theta y^2}\,dy$$

$$\int_t^\infty y f(y)\,dy = \frac{\theta}{(1+\theta)}\left[(1+2\theta)\int_t^\infty y^2 e^{-\theta y^2}\,dy + \theta\int_t^\infty y^4 e^{-\theta y^2}\,dy\right]$$

$$\int_t^\infty y f(y)\,dy = \frac{\theta}{(1+\theta)}\left[\frac{(1+2\theta)}{2\theta^{\frac{3}{2}}}\Gamma\left(\frac{3}{2},t^2\theta\right) + \frac{1}{2\theta^{\frac{3}{2}}}\Gamma\left(\frac{5}{2},t^2\theta\right)\right] \qquad (20)$$

Incorporating Eq. (20) into Eq. (19), we get the following expression

$$m(t) = \frac{1}{S(t)}\left\{\frac{1}{2(1+\theta)\theta^{\frac{1}{2}}}\left[(1+2\theta)\Gamma\left(\frac{3}{2},t^2\theta\right) + \Gamma\left(\frac{5}{2},t^2\theta\right)\right]\right\} - t. \qquad (21)$$

3.8. Order Statistics

The random variables $Y_{(i)}$, $Y_{(n)}$, and $Y_{(m)}$, represents order statistics of high, minimum, and maximum ranks. Below are PDFs for ith order statistics.

$$f_{(i,n)} = \frac{n!}{(i-1)!\,(n-i)!} g(y)[G(y)]^{i-1}[1-G(y)]^{n-i}$$

Using pdf and cdf of SZLD, the pdf of $Y_{i,n}$is given defined below:

$$f_{(i,n)} = \frac{n!}{(i-1)!\,(n-i)!}\left[1 - \left(1+\frac{\theta y^2}{2(1+\theta)}\right)e^{-\theta y^2}\right]^{i-1}\left[\left(1+\frac{\theta y^2}{2(1+\theta)}\right)e^{-\theta y^2}\right]^{n-i}\frac{\theta y(1+2\theta+\theta y^2)}{(1+\theta)}e^{-\theta y^2} \quad (22)$$

The pdf of the smallest order statistic is

$$f_{(1,n)} = \frac{n\theta}{(1+\theta)} y(1+2\theta+\theta y^2)e^{-\theta y^2}\left[\left(1+\frac{\theta y^2}{2(1+\theta)}\right)e^{-\theta y^2}\right]^{n-1} \quad (23)$$

The pdf of the largest order statistic is

$$f_{(n,n)} = \frac{n\theta}{(1+\theta)} y(1+2\theta+\theta y^2)e^{-\theta y^2}\left[1-\left(1+\frac{\theta y^2}{2(1+\theta)}\right)e^{-\theta y^2}\right]^{n-1} \quad (24)$$

4. Parameter Estimation and Simulation Study

In this section, the maximum likelihood (ML) estimate approach is used to determine the SZLD parameters. The mathematical formulas for the procedure are derived. Additionally, an analysis using a Monte Carlo simulation is performed for a range of sample sizes (n = 20, 50, 100, 150, and 500) and parameter combinations.

4.1. ML Estimation

Consider $y_1, y_2, y_3, \dots, y_n$ be a random sample of size n from SZLD. Then the likelihood function is subsequently obtained as follows

$$\prod_{i=1}^{n} f(\boldsymbol{y}) = \prod_{i=1}^{n}\left[\frac{\theta y_i\left(1+2\theta+\theta {y_i}^2\right)}{(1+\theta)}e^{-\theta {y_i}^2}\right] \quad (25)$$

The log-likelihood function is

$$l(L) = n\log(\theta) - n\log\big((1+\theta)\big) + \sum_{i=1}^{n}\log(y_i) + \sum_{i=1}^{n}\log(1+2\theta+\theta {y_i}^2) - \theta\sum_{i=1}^{n} {y_i}^2 \quad (26)$$

Now differentiate Eq. (26)with respect to parameter, respectively.

$$\frac{\partial l(L)}{\partial\theta} = \frac{n}{\theta(1+\theta)} + \sum_{i=1}^{n}\frac{2+{y_i}^2}{\left(1+2\theta+\theta {y_i}^2\right)} - \sum_{i=1}^{n} {y_i}^2 \quad (27)$$

Equation (27) been solved simultaneously in order to yield the ML estimations of the parameter, which equate to zero. However, the solutions to these equations have no closed form. Therefore, numerical methods are applied.

4.2. Simulation Study

The performance of the SZLD distribution is assessed using the N=5000 repetitions of the Monte Carlo simulation research. MSE, coverage probability (CP) of 95%, and absolute bias are used to elaborate on MLE performance. Different sets of actual parameter values are applied. For the simulation, sample sizes of n=15, 30, 80, 150, and 300 will be taken into account. The results of the simulation investigation are given in Table 3 below. The estimated values of parameters with absolute bias and MSE are obtained using simulated data. The findings indicate that bias and MSE are decreased with larger sample sizes.

Table 3. Simulation using MLEs for different values of θ.

Para.	n	Estimates	MSE	biais
		$\hat{\theta}$	$\hat{\theta}$	$\hat{\theta}$
$\theta = 0.10$	15	0.1119	1.4154 10^{-4}	1.1897 10^{-2}
	30	0.1000	8.0866 10^{-5}	8.9925 10^{-3}
	80	0.1021	4.5238 10^{-6}	2.100 10^{-3}
	150	0.1009	8.4092 10^{-7}	9.1701 10^{-4}
	300	0.1005	3.2694 10^{-7}	5.7179 10^{-4}
$\theta = 0.75$	15	0.7702	4.1032$10^{-4}$	2.0256 10^{-2}
	30	0.7652	2.3302$10^{-4}$	1.5265 10^{-2}
	80	0.7543	1.9042$10^{-5}$	4.3637$10^{-3}$
	150	0.7541	1.7336$10^{-5}$	4.1636$10^{-3}$
	300	0.7511	1.3448 10^{-6}	1.1596 10^{-3}
$\boldsymbol{\theta = 2}$	15	2.2131	4.5423$10^{-2}$	0.2131
	30	2.1051	1.1062$10^{-2}$	0.1051
	80	2.0401	1.6146 10^{-3}	4.0182 10^{-2}
	150	2.0230	5.3003 10^{-4}	2.3022 10^{-2}
	300	2.0064	4.1260 10^{-5}	6.4234 10^{-3}

5. Application of SZLD Distribution

In this section, we compare the SZLD to the baseline ZL distribution and some other renowned probability models available in the literature. We compare the fits with the following probability models; Zlindley (Saaidia et al.2024), two-parameter L1 (Shanker et al.2013), two-parameter L2 (Shanker et al.2013), quasi lindley(Shanker et al.2013) , new quasi lindley (Shanker et al.2013), gamma Lindley(Nadjar et al.2016), power Xlindley (B. Meriem et al.2022), XLindely (S. Chouia et al.2021), new Xlindley(N. Khodja *et al*.2023), Lindley (Lindley, D. V.1958) and Xgamma (S. Sen et al.2016) distributions. The model selection is carried out using the following statistics:

$$\mathrm{AIC} = -2\mathrm{L} + 2\mathrm{p}, \quad \mathrm{BIC} = -2\mathrm{L} + \mathrm{plog(n)}, \quad \mathrm{CAIC} = -2\mathrm{L} + \frac{2\mathrm{pn}}{\mathrm{n} - \mathrm{p} - 1}$$

5.1. Data Set I:

For DDT in Kale data

A numerical vector including 15 measurements of the pesticide DDT in kale, measured using diverse pesticide residue measuring methods and expressed in ppm (parts per million). The observations are: 2.79, 2.93, 3.22, 3.78, 3.22, 3.38, 3.18, 3.33, 3.34, 3.06, 3.07, 3.56, 3.08, 4.64, 3.34 (see Finsterwalder (1976)).

Table 4. The ML estimates,-2 log-likelihood, AIC, BIC, and CAIC for Data Set I

Model	density	$\hat{\theta}$	$\hat{\gamma}$	AIC	BIC	-2L	CAIC
two-parameter L1	$\frac{\theta^2(\gamma+x)e^{-\theta x}}{\gamma\theta+1}$	0.622391	0.0008552	58.74148	60.15758	54.74148	59.74148
gamma Lindley	$\frac{\theta^2((\gamma+\gamma\theta-\theta)x+1)e^{-\theta x}}{\gamma(\theta+1)}$	0.598546	20.45021	58.96757	60.38367	54.96757	59.96757
quasi Lindley	$\frac{\theta(\gamma+x\theta)e^{-\theta x}}{\gamma+1}$	0.89774	0.0006939	67.49544	68.91155	63.49544	68.49544
new quasi Lindley	$\frac{\theta^2(\theta+\gamma x)e^{-\theta x}}{\gamma+\theta^2}$	0.593307	24.60633	58.90761	60.32371	54.90761	59.90761
two parameter L2	$\frac{\theta^2(1+\gamma x)e^{-\theta x}}{\gamma+\theta}$	0.595902	23.3598	59.06532	60.48142	55.06532	60.06532
Power XLindley	$\frac{\alpha\theta^2(2+\theta+x^{\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}}{(\theta+1)^2}$	1.44965	0.381104	106.8118	108.2279	102.8118	107.8118
ZLindley	$\frac{\theta(1+2\theta+\theta x)e^{-\theta x}}{2(\theta+1)}$	0.4190237	/	66.06775	66.7758	64.06775	66.37544
Square ZLindley	$\frac{\theta x(1+2\theta+\theta x^2)e^{-\theta x^2}}{(\theta+1)}$	**0.1346802**	**/**	**45. 115**	**45. 823**	**43. 115**	**45. 423**

5.2. Data Set II:

The second data set is illustrated below and is utilized by 14. The figures collected indicate the breaking stress of carbon fibers 50mm in length (GPa). The observations are: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12,2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85,2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31,3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90 (See Staudte and Sheather (1990)).

Table 5. The ML estimates,-2 log-likelihood, AIC, BIC, and CAIC for Data Set II.

Model	**density**	θ	γ	**AIC**	**BIC**	**-2L**	**CAIC**
two-parameter L1	$\frac{\theta^2(\gamma+x)e^{-\theta x}}{\gamma\theta+1}$	0.73366	0.0005	228.0497	232.429	224.0497	228.2402
gamma Lindley	$\frac{\theta^2((\gamma+\gamma\theta-\theta)x+1)e^{-\theta x}}{\gamma(\theta+1)}$	0.7201	54.8936	228.413	232.792	224.413	228.604
quasi Lindley	$\frac{\theta(\gamma+x\theta)e^{-\theta x}}{\gamma+1}$	1.0899	0.000302	241.962	246.341	237.962	242.153
new quasi Lindley	$\frac{\theta^2(\theta+\gamma x)e^{-\theta x}}{\gamma+\theta^2}$	0.7217	54.88575	228.505	232.884	224.505	228.695
two parameter L2	$\frac{\theta^2(1+\gamma x)e^{-\theta x}}{\gamma+\theta}$	0.7226	71.85723	228.534	232.914	224.534	228.725
Power XLindley	$\frac{\alpha\theta^2(2+\theta+x^{\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}}{(\theta+1)^2}$	1.5686	0.39675	423.317	427.697	419.317	423.508
ZLindley	$\frac{\theta(1+2\theta+\theta x)e^{-\theta x}}{2(\theta+1)}$	0.4948	/	261.059	263.248	259.059	261.121
Zeghdoudi	$\frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta}$	0.9688	/	215.452	217.641	213.452	215.514
XLindley	$\frac{\theta^2(2+\theta+x)e^{-\theta x}}{(1+\theta)^2}$	0.5149	/	256.929	259.118	254.929	256.991
Exponential	$\theta e^{-\theta x}$	0.3624	/	267.988	270.178	265.988	268.051
New-XLindley	$\frac{\theta(1+\theta x)e^{-\theta x}}{2}$	0.5787	/	253.322	255.512	251.322	253.385
Lindley	$\frac{\theta^2(1+x)e^{-\theta x}}{(1+\theta)}$	0.5902	/	246.768	248.957	244.768	246.831
Xgamma	$\frac{\theta^2\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{1+\theta}$	0.8211	/	249.439	251.628	247.439	249.5014
Square ZLindley	$\frac{\theta x(1+2\theta+\theta x^2)e^{-\theta x^2}}{(\theta+1)}$	**0.1742**	**/**	**191. 421**	**192. 132**	**189. 421**	**191. 732**

6. Conclusion

A novel one-parameter distribution that is proposed and studied is called the "Square ZLindley distribution." Among the mathematically derived properties of SZLD are moments and related metrics. In addition, reliability parameters, hazard function, and mean residual life are obtained. The estimation of the SZLD parameters is done using the well-known likelihood technique. The generated MLEs are assessed using an extensive simulation study. Towards the end of this paper, the applicability of the new model is demonstrated using two real datasets.Out of a few popular distributions, the SZLD provides the best match. Potential directions for future research include examining maximum likelihood functions under various censoring schemes to increase the adaptability of the Square ZLindley hybrid model. The Square ZLindley hybrid distribution's parameter estimation could also be improved by investigating Bayesian estimation methods that include appropriate priors and risk functions.

Declaration of Author Contributions

The authors declare that they have contributed equally to the article. All authors declare that they have seen/read and approved the final version of the article ready for publication.

Declaration of Conflicts of Interest

All authors declare that there is no conflict of interest related to this article.

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